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A RESIDUAL METHOD OF FINITE DIFFERENCING FOR THE ELLIPTIC TRANSPORT PROBLEM AND ITS APPLICATION TO CAVITY FLOW

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SUMMARY

A residual method of finite differencing the governing differential equation for the elliptic transport problem is presented. The new finite differencing technique is applied to (1) the one-dimensional transport problem and (2) the cavity flow problem for numerical illustrations. The results indicate the validity of the residual method of finite differencing. The usual method of term-by-term finite differencing, and considerations such as central differencing, hybrid differencing and upwind differencing are not needed in the present residual method.

KEY WORDS Finite Difference Residual Method Elliptic Equation Cavity Flow

INTRODUCTION

There are two well-known methods of deriving finite difference equations from differential ones: by way of Taylor-series expansion and by integration over finite areas, together with assumptions about the distribution of the variables between the nodes of the grid. In the present work another type of finite differencing scheme, known as the residual scheme, is presented to solve the elliptic transport problem. Various schemes such as the central (CDS), upwind (UDS), hybrid (HDS), skew or quadratic upwind differencing schemes[1–3] have been developed and investigated for such problems. The finite difference equation obtained by the residual approach has been compared with those obtained by CDS, UDS and HDS for the one-dimensional problem. As a numerical illustration for the two-dimensional elliptic problem, the classical problem of flow in a square cavity has been chosen. The results are compared with those of Burggraf[4] who has given extensive finite difference results for low and moderate Reynolds numbers. In this context it may be worthwhile to mention that in an earlier work the authors[5] applied the residual approach to solve the parabolic transport problem and found encouraging results.

It should be mentioned that the residual method proposed independently by the authors appears similar to the finite analytic method by Chen and Chen.[6] However, there are conceptual and procedural differences between the two methods which make the residual method more versatile. Specifically, the present method provides algebraic expressions which correspond to the differential expressions in the governing equations both in sign and magnitude. Thus, the residual method allows the physical interpretation and logical extension of the finite difference expression for simple cases to complicated cases without recourse to the residual method of formulation in

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each case. Also, the residual method can be applied separately to various groups formed out of the complete governing differential equation, and then the residual expressions may be added up to obtain the formulation for the complete governing equation. This is a simple but approximate procedure. The two-dimensional formulation (15) in the present text has been carried out in this manner, and a logical extension of the present one-dimensional formulation (7) including effectively the variation of the coefficients in the governing equation (1) is reported elsewhere.[7]

FORMULATION

One-D problem

The starting point of the residual method of finite differencing is that an incorrect set of trial values of the unknowns at the various grid nodes satisfy the governing differential equation with an additional residue. The differential equation with the additional residue is then solved in the cell in such a manner as to match the trial values at the nodes in the cell. This procedure provides an algebraic equation for the residue which is equated to zero to obtain the finite difference equation.

The differential equation for the one-dimensional elliptic problem involving transport of a quantity ϕ by convection, diffusion and source term in a small cell may be written as

$$\rho u \mathrm{d}\phi/\mathrm{d}x - \Gamma \mathrm{d}^2\phi/\mathrm{d}x^2 - s_\phi = 0 \tag{1}$$

It is assumed that the local velocity u, the density ρ, the diffusivity Γ and the source term s_ϕ are constant in the small cell. The nodes E, P and W lie along the x-direction in Figure 1.

For incorrect trial values ϕ_E, ϕ_P and ϕ_W, one may write a residual equation as

$$\rho u \mathrm{d}\phi/\mathrm{d}x - \Gamma \mathrm{d}^2\phi/\mathrm{d}x^2 - s_\phi = R \tag{2}$$

where R stands for a residue in the cell. A solution of equation (2) is of the following form.

$$\phi = C_0 + C_1 x + C_2 e^{\rho u x/\Gamma} \tag{3}$$

with

$$\rho u C_1 = R \tag{4}$$

The values ϕ_E, ϕ_P and ϕ_W at $x = \Delta x, 0, -\Delta x$ are used in equation (3) to determine the unknown

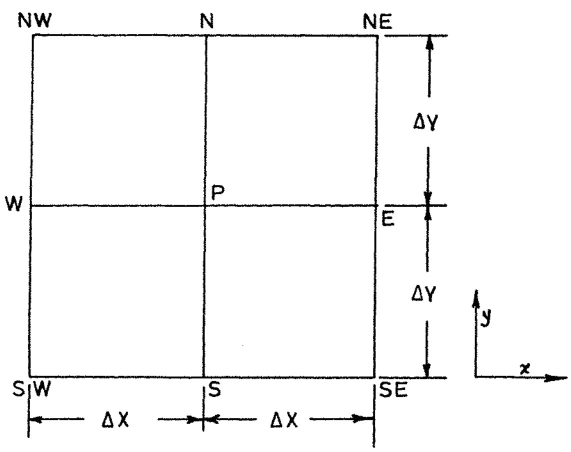

Figure 1. Elliptic cell

coefficients C_0, C_1 and C_2. The expression for R is then derived as

$$R = \rho u(\phi_E - \phi_W)/(2\Delta x) - \Gamma f(P_x)(\phi_E + \phi_W - 2\phi_P)/\Delta x^2 - s_\phi \quad (5)$$

where

$$P_x = \rho u \Delta x/\Gamma, f(P) = P/2(e^P - e^{-P})/(e^P + e^{-P} - 2) \quad (6)$$

The finite difference equation is now obtained by equating R to zero as follows:

$$\rho u(\phi_E - \phi_W)/(2\Delta x) - \Gamma f(P_x)(\phi_E + \phi_W - 2\phi_P)/\Delta x^2 - s_\phi = 0 \quad (7)$$

In equation (6), P_x stands for the local grid Peclet number and the factor f depends solely on this number.

Unlike conventional methods of finite differencing the differential equation term by term, the residual approach provides an F.D.E. for the whole differential equation. The final F.D.E. (7) can, however, be identified term by term with the governing differential equation in the present case. In this case, the convection term and the diffusion term are both in CDS form. However, the diffusion term has a factor f which depends on the grid Peclet number. This makes the present F.D.E. substantially different from conventional schemes. As the results in the illustration will show, the F.D.E. (7) provides both stability and accuracy of numerical solution.

Two-D problem

The governing equation for the two-dimensional elliptic transport problem for a small cell may be written as

$$\rho u \partial\phi/\partial x + \rho v \partial\phi/\partial y - \Gamma \partial^2\phi/\partial x^2 - \Gamma \partial^2\phi/\partial y^2 - s_\phi = 0 \quad (8)$$

where velocities, physical properties and source are taken as constants in the small cell. The elliptic cell is shown in Figure 1.

Following the formulation for the one-dimensional case, the residual equation may be written as

$$\rho u \partial\phi/\partial x + \rho v \partial\phi/\partial y - \Gamma \partial^2\phi/\partial x^2 - \Gamma \partial^2\phi/\partial y^2 - s_\phi = R \quad (9)$$

One now requires a local solution to equation (9). With nine trial values of ϕ at the nodes of the cell, nine independent simple relations with unknown coefficients could be made. Solving these equations, one can obtain a rigorous finite difference equation for the problem. It is possible in principle, but difficult in practice. Moreover, extension to three dimensions may be extremely difficult.

For this reason, a simpler method, based on group-wise formulation is used to derive F.D.E. for two dimensional cases as follows,

Equation (9) is arranged by grouping terms as given below

$$R = [W(R_{ex} + R_{ey}) + (1 - W)(R_{px} + R_{py})] - s_\phi \quad (10)$$

where R_{ex} and R_{ey} are elliptic combinations in the x and y directions, respectively,

$$R_{ex} = \rho u \partial\phi/\partial x - \Gamma \partial^2\phi/\partial x^2; \; R_{ey} = \rho v \partial\phi/\partial y - \Gamma \partial^2\phi/\partial y^2 \quad (11a, b)$$

and R_{px}, R_{py} are parabolic combinations in the x and y directions, respectively,

$$R_{px} = \rho u \partial\phi/\partial x - \Gamma \partial^2\phi/\partial y^2; \; R_{py} = \rho v \partial\phi/\partial y - \Gamma \partial^2\phi/\partial x^2 \quad (12a, b)$$

and W is a weight coefficient whose value may be chosen between 0 and 1 to emphasize parabolic or elliptic natures. The total residue R is thus broken up into residues R_{ex}, R_{ey}, R_{px} and R_{py} which are simple to evaluate separately. In fact, the finite difference form for elliptic equation (11a) follows directly from equation (5). Similar expression for R_{ey} can be written.

The detail of the finite difference formulation for the parabolic equation (12a, b) is reported elsewhere.[5] The finite difference cell for equation (12a) will be either the left half or the right half of the full elliptic cell in Figure 1. Similarly, the finite difference cell for equation (12b) will be either the top half or the bottom half of the full cell. It depends on the directions of the velocities. The results are presented below.

$$u > 0: R_{px} = \rho u(\phi_P - \phi_W)/\Delta x - \Gamma/\Delta y^2[\theta_x(\phi_{NW} - 2\phi_W + \phi_{SW}) + (1 - \theta_x)(\phi_N - 2\phi_P + \phi_S)] \quad (13a)$$

$$u < 0: R_{px} = \rho u(\phi_E - \phi_P)/\Delta x - \Gamma/\Delta y^2[\theta_x(\phi_{NE} - 2\phi_E + \phi_{SE}) + (1 - \theta_x)(\phi_N - 2\phi_P + \phi_S)] \quad (13b)$$

$$u = 0: R_{px} = -\Gamma/\Delta y^2(\phi_N - 2\phi_P + \phi_S) \quad (13c)$$

where

$$F_x = \Gamma\Delta x/(\rho|u|\Delta y^2) = (\Delta x/\Delta y)^2/|P_x| \quad (14a)$$

$$\theta_x = \theta(F_x) = \frac{2F_x(1 - \cos Z) - (1 - e^{-Z^2F_x})}{2F_x(1 - e^{-Z^2F_x})(1 - \cos Z)}; \quad Z = \sqrt{2} \quad (14b, c)$$

Similar equations for R_{py} can be written.

Previous works consider only elliptic combinations namely R_{ex} and R_{ey} for the finite difference formulation of equation (8). In the present scheme both elliptic and parabolic combinations have been considered. Whether such consideration will provide an improved method requires numerical testing. The finite difference formulations for R_{ex}, R_{ey}, R_{px} and R_{py} have already been described. These can be used in equation (10) to obtain the total residue. When this residue is equated to zero one can obtain the finite difference equation as

$$[W(R_{ex} + R_{ey}) + (1 - W)(R_{px} + R_{py})] - s_\phi = 0 \quad (15)$$

So far, the coefficients ρu, ρv, Γ in the differential equation (8) have been treated as constants in the cell. As the cell is usually small compared to the global region, this is approximately applicable even when these vary in the cell, provided one uses average values in the cell. Investigation on cavity flow shows that the present F.D.E. is applicable also in the case where ϕ is u or v in equation (8).

ILLUSTRATIONS

One-D problem

The governing equation (16a, b), boundary condition (16c) and the exact solution (17) for a simple one-dimensional elliptic problem with variable velocity are presented below.

$$\rho u \mathrm{d}\phi/\mathrm{d}x - \Gamma \mathrm{d}^2\phi/\mathrm{d}x^2 = 0, \quad 0{\cdot}1 < x < 1 \quad (16a)$$

$$\rho = 1, \quad \Gamma = 1, \quad u = M/x \quad (M > 0) \quad (16b)$$

$$\phi(0{\cdot}1) = (0{\cdot}1)^{M+1}, \quad \phi(1) = 1 \quad (16c)$$

$$\phi(x) = x^{M+1}, \quad 0{\cdot}1 \leqslant x \leqslant 1 \quad (17)$$

The density ρ and diffusivity Γ are chosen as unity for convenience. The velocity u is high upstream and it decreases downstream. In the above equations M is a parameter. The exact solution (17) depends strongly on this parameter. The equations have some bearing on the physical problem of

Table I. Values of ϕ at $x = 0{\cdot}9$ for equation (16)

M Scheme*	4 0·4–4	8 0·8–8	12 1·2–12	16 1·6–16	20 2–20
Exact	0·590	0·387	0·254	0·167	0·109
Present	0·587	0·383	0·249	0·161	0·104
CDS	0·581	0·354	0·185	0·053	−0·053
UDS	0·641	0·498	0·407	0·345	0·299
HDS	0·581	0·354	0·185	0·053	0·000

* Entries indicate range of grid Peclet numbers

the one-dimensional distribution of ϕ in a fluid flowing through a porous channel with controlled suction.

A numerical solution of the problem considering nine uniform divisions has been carried out according to CDS, UDS, HDS and the present F.D.E. (7). An average local velocity in a cell given by

$$u_{av} = (u_W + 2u_P + u_E)/4 \tag{18}$$

is used in all the cases for finite differencing so as to compare these schemes on the same basis. The numerical results for ϕ at $x = 0{\cdot}9$ are presented in Table I along with the exact values. All the results are rounded up to three decimal places. The grid Peclet numbers for a particular M vary between $M/10$ and M. These ranges are also shown in Table I. Large values of M generally indicate large values of grid Peclet numbers in this problem.

It is observed that the present scheme provides the most accurate results among the various schemes for both small and large grid Peclet numbers. Those using HDS are the next best. The tabulated results for CDS look as good as HDS but negative values of ϕ at certain values of x are predicted wrongly by CDS for large grid Peclet numbers. The results using UDS are less accurate compared to other schemes for this problem.

Two-D cavity problem

Flow in a square cavity induced by movement of a plate has been investigated by various authors.[4,8–10] A numerical solution of this problem by the present two dimensional scheme is

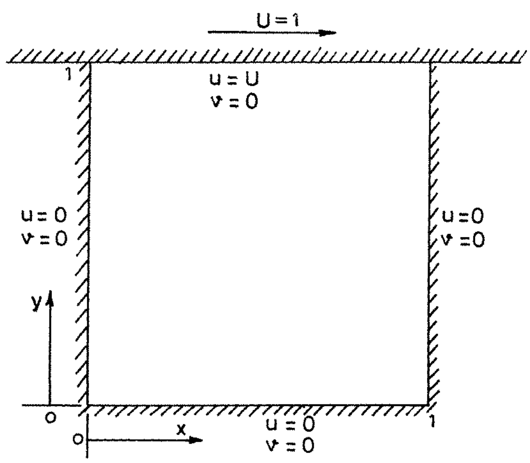

Figure 2. Geometry and boundary conditions

presented here. The geometry of the problem and the co-ordinate system used are given in Figure 2. The boundary conditions have been indicated in the sketch. The equations governing the flow are as follows:

$$\partial u/\partial x + \partial v/\partial y = 0 \tag{19}$$

$$\rho u \partial u/\partial x + \rho v \partial u/\partial y = -\partial p/\partial x + \mu(\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2) \tag{20}$$

$$\rho u \partial v/\partial x + \rho v \partial v/\partial y = -\partial p/\partial y + \mu(\partial^2 v/\partial x^2 + \partial^2 v/\partial y^2) \tag{21}$$

The momentum equations (20) and (21) are finite differenced according to equation (15), as described earlier. The continuity equation (19) is used for pressure correction as described by Patankar and Spalding.[11]

In the present scheme of solution, a pressure in cell arrangement is considered as shown in Figure 3. Each small cell is treated for continuity and the velocity locations are across the faces of each such cell. The u-momentum, v-momentum and continuity cells are shown separately in Figures 4(a), (b) and (c). The coefficients of equations (20) and (21) are averaged in the cell for finite differencing and the expressions for average u and v required for this are written in the Figures as well. The physical properties μ and ρ are constant in the present problem and no averaging is needed.

The finite difference equations are solved as per the following schedule:

(i) Initial trial values of u, v and p are selected.

(ii) The finite difference equations for u-momentum and v-momentum are solved for by successive substitution.

(iii) The pressure field is corrected to satisfy continuity in each cell. It may be noted that corrections to pressure gradients rectify velocities across each face of the continuity cell and their relations are obtained from the F.D.E. for the momentum equations. No conditions for pressure correction are needed near the boundaries; rather the condition of no flow across the boundaries is automatically imposed for boundary cells.

(iv) The velocities are then corrected to take into account the improved pressure field.

(v) Steps (ii) to (iv) are repeated until the residues are reduced to very small quantities.

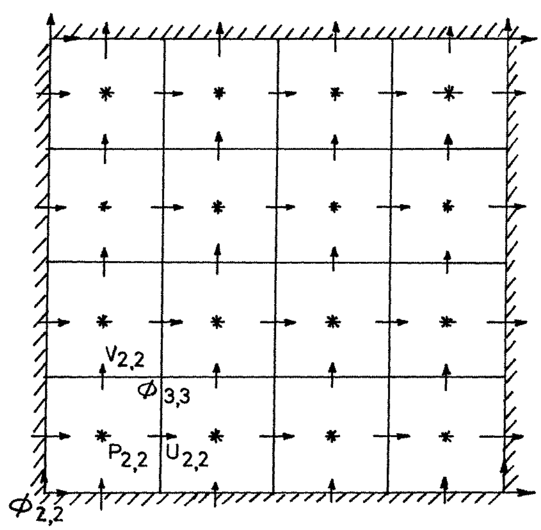

Figure 3. Grid arrangement

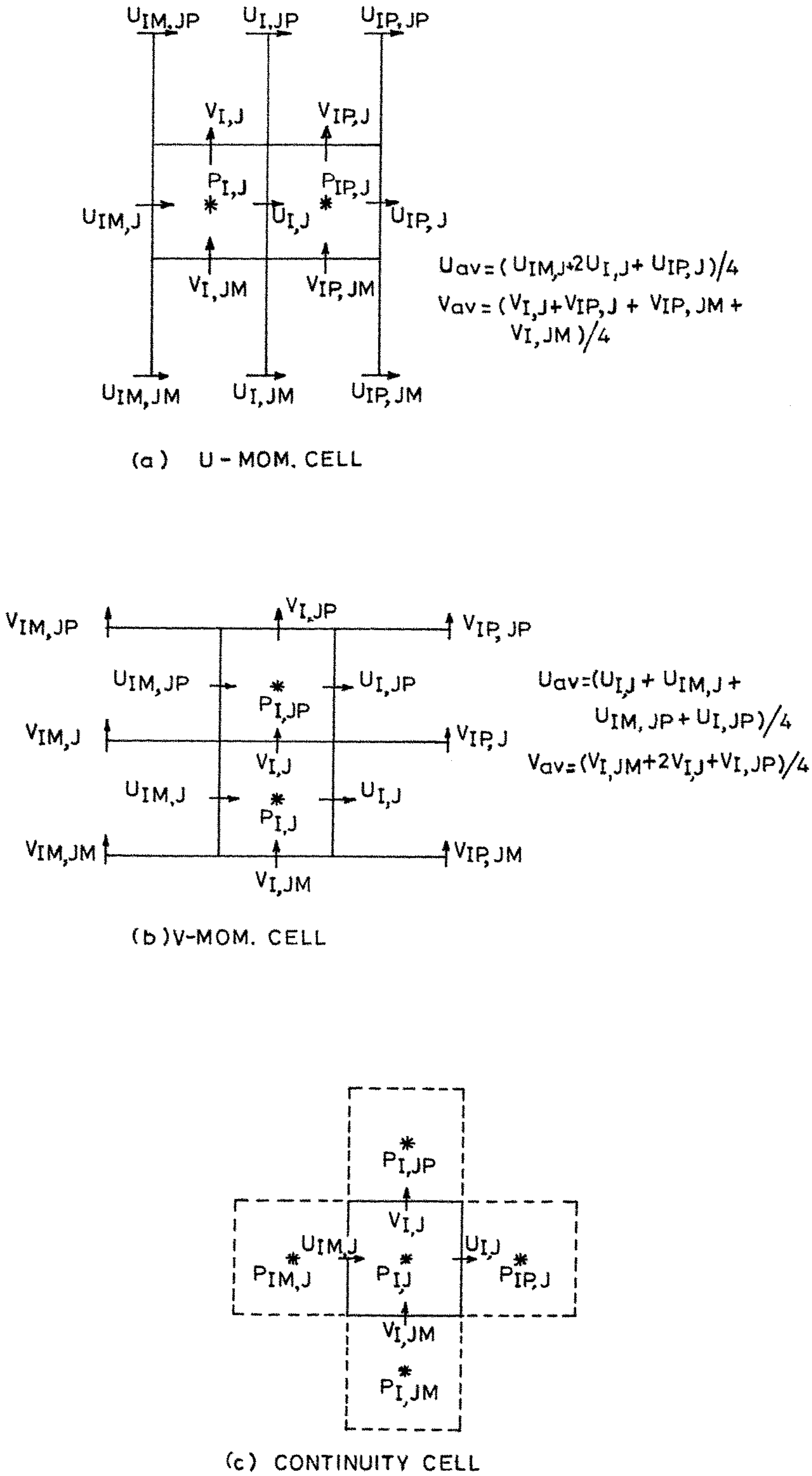

Figure 4. Momentum and continuity cells

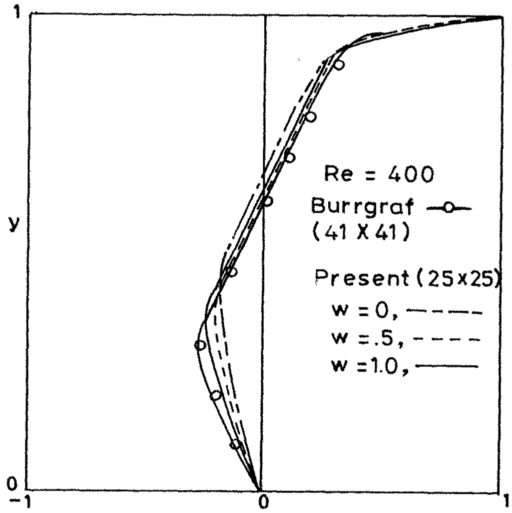

Figure 5. Central line u-velocity for different W

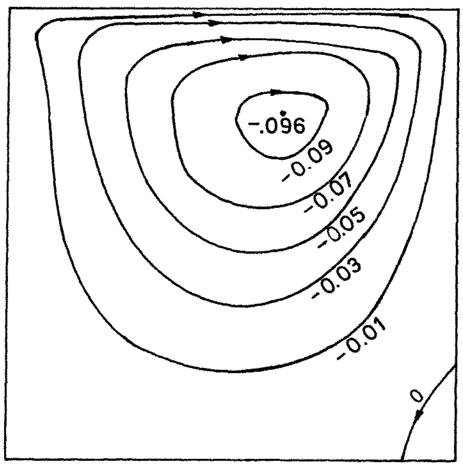

Figure 6(a). Streamlines for $Re = 100$ (16×16 mesh)

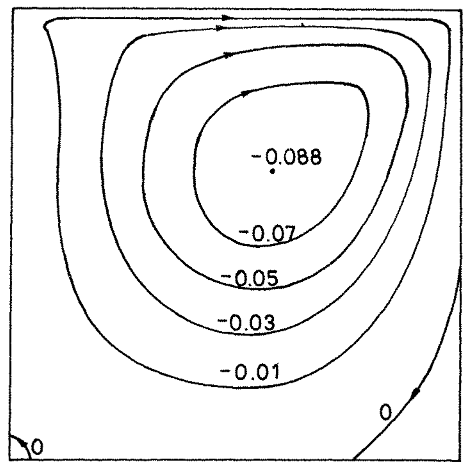

Figure 6(b). Streamlines for $Re = 400$ (25×25 mesh)

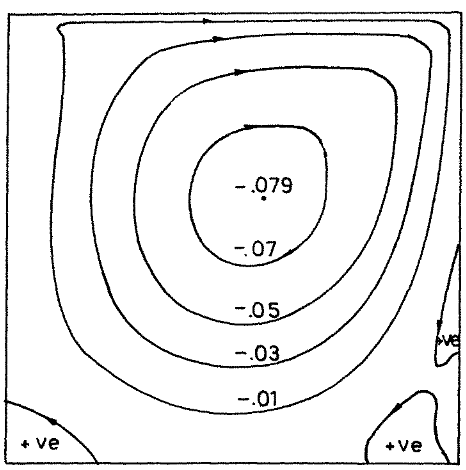

Figure 6(c). Streamlines for $Re = 1000$ (41×41 mesh)

RESULTS

Computations have been carried out for Reynolds numbers 100, 400 and 1000. The effect of grid size has been investigated considering (16 × 16) and (25 × 25) meshes. Qualitatively the effect of grid size is insignificant. However, quantitatively the results differ. For example, the peak value of the vertical central line *u*-velocity near the bottom for $Re = 400$ differs by 22 per cent when the grid is changed from (16 × 16) to (25 × 25).

The weight W, which has been introduced in the finite difference equation (10) to emphasize parabolic or elliptic combination, has also been investigated for $Re = 400$. The problem has been solved using $W = 0, 0{\cdot}5$ and 1. The values $W = 0$ and $W = 1$ indicate fully parabolic and fully elliptic combinations, respectively, and $W = 0{\cdot}5$ indicates a parabolic–elliptic combination. No problem of stability or convergence has occurred in these cases. The *u*-velocity along the vertical central line for these three cases is plotted and compared with that of Burggraf[4] in Figure 5. From Figure 5, it can be seen that the plot for $W = 1$ is closest to Burggraf's result. This is expected since the global problem is elliptic. On the basis of this, the rest of the calculations have been performed with $W = 1$.

The streamline patterns for different Reynolds numbers are shown in Figures 6(a), (b) and (c). It can be seen that the primary vortex centre moves downwards with increased Reynolds number. A feature of two small secondary cells near right lower corner is observed in the present case for $Re = 1000$. A comparison of the present values of stream function at vortex centre with those of Burggraf is presented in Table II. It may be remarked that for $Re = 1000$, Burggraf[4] could not give any solution, but the present method yields a solution. Plots of vertical central line velocity for $Re = 100$, 400 and 1000 are shown in Figure 7. It may be emphasized that the present illustrations are based on the approximation of a single average velocity in a cell for finite differencing, and the results are encouraging even with this gross approximation.

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Table II. Stream function at the centre of primary vortex for two dimensional cavity flow problem

Re	Mesh	Present	Burggraf
100	11 × 11	0·0815	0·0784
	16 × 16	0·0961	—
	21 × 21	—	0·0955
	41 × 41	—	0·1015
400	16 × 16	0·0741	—
	21 × 21	—	0·0675
	25 × 25	0·0881	—
	41 × 41	—	0·1017
1000	25 × 25	0·0682	no solution
	41 × 41	0·0791	no solution

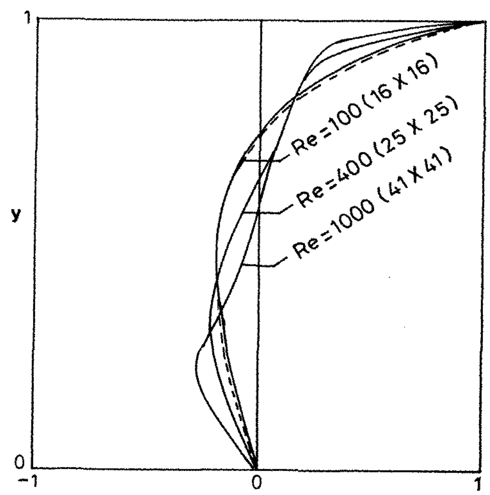

Figure 7. Central line *u*-velocity for different $Re(W = 1)$

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